The Heat Kernel on AdS_5 and its Applications

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Reference: Rajesh Gopakumar, Rajesh K. Gupta and S.L., to appear

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Introduction

- The one-loop partition function contains the leading quantum corrections to a classical background.
- It is also potentially tractable.
- It decomposes into separate contributions from every particle in the spectrum of the theory, and is sensitive only to that part of the Lagrangian which is quadratic in the fluctuations.
- If Gravity is the only background field, then the contribution of a particle to the one-loop partition function is $ln\,det\left(\Delta_{(s)} + m^2\right)$ where $\Delta_{(s)}$ is the Laplacian for a spin-s particle.
- We will define this determinant through the heat kernel and evaluate it for the thermal quotient of AdS_5.

Motivations

- This program has already yielded insights into 3-d gravity.
- The heat kernel on AdS_3 has been calculated in this manner and the one-loop partition function of ($\mathcal{N} = 1$ super-)gravity has been shown to be the vacuum character of the ($\mathcal{N} = 1$ super-)Virasoro algebra.

  [Giombi, Maloney, Yin arXiv:0804.1773]

  [David, Gaberdiel,Gopakumar arXiv:0911.5085]
- This has also provided evidence that topologically massive gravity at its chiral point is dual to a *log*-CFT.

  [Grumiller, Vassilevich, Gaberdiel arXiv:1007.5189]
- The one-loop partition function of higher spin theories on AdS_3 has been shown to be determined in terms of the vacuum characters of $\mathcal{W}_n$ algebras.

  [Gaberdiel,Gopakumar,Saha arXiv:1009.6087]
- Further, this has lead to a proposal for the CFT dual of these higher spin theories.

  [Gaberdiel,Gopakumar arXiv:1011.2986]

The Heat Kernel

- We will now define the determinant of the laplacian using the property that $\ln det\left(-\Delta_{(s)}\right) = \mathrm{Tr}\ln\left(-\Delta_{(s)}\right)$, and the relation that $\ln\left(-\Delta_{(s)}\right) = -\int_0^\infty \frac{dt}{t} e^{t\Delta_{(s)}}$.
- This leads us to define

$$\ln det\left(-\Delta_{(s)}\right) = -\mathrm{Tr}\int_0^\infty \frac{dt}{t} e^{t\Delta_{(s)}} = -\int_0^\infty \frac{dt}{t} \mathrm{Tr} e^{t\Delta_{(s)}}.$$

Now the trace is given by

$$\begin{aligned} \mathrm{Tr} e^{t\Delta} &\equiv \int \sqrt{g} d^n x \sum_a \langle x, a | e^{t\Delta} | x, a \rangle \\ &= \int \sqrt{g} d^n x \sum_a \sum_n \psi_{n,a}(x) \psi_{n,a}(x)^* e^{tE_n}, \end{aligned}$$

where $\psi_{n,a}(x)$ are eigenfunctions of the Laplacian, and a is a local Lorentz index.

The Heat Kernel

- The expression above is just the trace of the heat kernel defined as

$$K_{a,b}\left(x,y,t\right)=\sum_{n}\psi_{n,a}\left(x\right)\psi_{n,b}\left(y\right)^{*}e^{tE_{n}},$$

  where the trace was done over both the local Lorentz index a, as well as the continuous space-time index x.
- To calculate the one-loop partition function we therefore need to know
  - the eigenvalues E_n of the Laplacian
  - its eigenfunctions $\psi_{n,a}\left(x\right)$
  - how to do the sum over n and the trace over a and x.
- We will see that because AdS_5 is a symmetric space, there is a group theoretic structure underlying all three of these questions, which we can exploit to evaluate the traced heat kernel for particles with arbitrary spin.

A toy model: The scalar on S^2

- As an illustration of the techniques that are involved, we evaluate the heat kernel for a scalar on S^2.
- The eigenvalues of the Laplacian are $\ell(\ell+1)$, which is the quadratic Casimir of the spin-ℓ representation of $SU(2)$.
- The corresponding eigenfunctions are the $Y_{\ell m}(\theta,\phi)$, which are elements of the spin-ℓ Wigner matrices.

$$Y_{\ell m}(\theta,\phi)=\sqrt{\frac{2\ell+1}{4\pi}}D^{\ell}_{0,m}\left(\mathcal{R}(\phi,\theta,0)^{-1}\right)$$

- The sum over the degenerate eigenvalues can be performed by using the addition theorem

$$\sum_{m=-\ell}^{\ell}Y_{\ell}^{m}(\hat{x})\,Y_{\ell}^{m}(\hat{x})^{*}=\frac{2\ell+1}{4\pi},$$

which is a consequence of the group multiplication law of Wigner Matrices.

- The heat kernel is then given by

$$\begin{aligned} K\left(\hat{x}_1, \hat{x}_2, t\right) &= \sum_{\ell,m} \frac{2\ell+1}{4\pi} D^{\ell}_{0,m}\left(\mathcal{R}_1^{-1}\right) D^{\ell}_{m,0}\left(\mathcal{R}_2\right) e^{-tE_\ell} \\ &= \sum_{\ell} \frac{2\ell+1}{4\pi} D^{\ell}_{0,0}\left(\mathcal{R}_1^{-1}\mathcal{R}_2\right) e^{-tE_\ell} \end{aligned}$$

- We recognize the prefactor $\frac{2\ell+1}{4\pi}$ as $\frac{d_\ell}{d_0}\frac{1}{V_{S^2}}$, where d_ℓ is the dimension of the spin-ℓ representation of $SU(2)$, $d_0 = 1$ is the dimension of the singlet representation of $U(1)$.
- This is (almost) the answer for an arbitrary-spin particle on a generic symmetric space!

- We can express these results in a more "group theoretic" manner which anticipates the more general theory that we'll use.
- The scalar transforms under the $s = 0$ representation of $U(1)$ generated by J_3.
- The eigenfunctions of the Laplacian are unitary representation matrices of $SU(2)$ that transform in any spin-ℓ that contains the representation $s = 0$ of the $U(1)$.
- These representation matrices are functions of an $SU(2)$ element which depend on the coordinates on $S^2 \simeq SU(2)/U(1)$.
- The corresponding eigenvalue is the difference between the quadratic Casimir of the spin-ℓ representation of $SU(2)$ and the quadratic Casimir of the $s = 0$ representation of the $U(1)$.

Strategy

- AdS_5 is realised as $SO(5,1)/SO(5)$, which is a symmetric space. Harmonic analysis on such spaces is well understood, and in principle one can carry out the above program directly for AdS_5.
- It is still simpler to consider $S^5 \equiv SO(6)/SO(5)$, and analytically continue our answers to the AdS_5 case.
- We will carry out this procedure for the thermal quotient of AdS_5.
- Our analysis is quite general and should be extendable to other quotients and symmetric spaces. It extends directly to AdS_{2n+1}.

Review: Harmonic Analysis on Homogeneous Spaces

- Fields are sections over G/H and are in one-to-one correspondence with irreps of H.
- Consider a particle transforming in some representation S of H, or a "spin-S" particle.
- If R is a representation of G such that it contains S when restricted to H, then the eigenvalue of the Laplacian is given by

$$\boxed{-E_R^{(S)} = C_2\left(R\right) - C_2\left(S\right).}$$

- Further, if $\sigma\left(x\right) \in G$ is a section over G/H, then the corresponding eigenfunctions are

$$\boxed{\mathcal{U}^{(R)}\left(\sigma\left(x\right)^{-1}\right)_a^{\ I} \equiv \langle a, S|\mathcal{U}^{(R)}\left(\sigma\left(x\right)^{-1}\right)|I\rangle}$$

The Heat Kernel

- The Heat Kernel for a spin-S particle is then

$$\boxed{K^S_{a,b}(x,y,t) = \sum_R \frac{d_R}{d_S}\frac{1}{V_{G/H}}\mathcal{U}^{(R)}\left(\sigma(x)^{-1}\sigma(y)\right)_a^{\ b} e^{tE_R^{(S)}},}$$

which is analogous to the expression on S^2.

- The group multiplication used here is the "addition theorem" for harmonics on generic symmetric spaces.
- We can also trace over the H indices to define

$$\boxed{K^S(x,y,t) = \sum_R \frac{d_R}{d_S}\frac{1}{V_{G/H}}\mathrm{Tr}_S\,\mathcal{U}^{(R)}\left(\sigma(x)^{-1}\sigma(y)\right) e^{tE_R^{(S)}}}$$

Harmonic Analysis on Quotients of G/H

- Consider an abelian discrete group Γ, generated by γ.
- We will consider the quotient space $\Gamma\backslash G/H$.
- If G/H is compact, $\Gamma \simeq \mathbb{Z}_N$.
- The trace over the heat kernel can be implemented when the choice of section is compatible with the quotienting.
- *i.e.* we must choose a section σ such that

$$\sigma\left(\gamma\left(x\right)\right) = \gamma \cdot \sigma\left(x\right).$$

- The other ingredient we need is the method of images.

The method of Images

Giombi, Maloney, Yin arXiv:0804.4589

- We can use the method of images to evaluate the heat kernel over quotients of G/H, *e.g.* AdS.
- The heat kernel on the quotient space $\Gamma\backslash G/H$ is given by

$$\boxed{K_{\Gamma}^{(S)}(x,y,t)=\sum_{\gamma\in\Gamma}K^{(S)}(x,\gamma(y),t)}$$

- The heat kernel on G/H satisfies the heat equation and

$$K_{ab}^{(S)}(x,y,0)=\delta(x-y)\,\delta_{ab}$$

- On $\Gamma\backslash G/H$, this boundary condition gets modified to

$$K_{ab}^{(S)}(x,y,0)=\sum_{\gamma\in\Gamma}\delta(x-\gamma(y))\,\delta_{ab}$$

- The solution on $\Gamma\backslash G/H$ then follows.

The thermal quotient of S^5

- Let us consider the unit S^5, which can be realised as the surface $|z_1|^2 + |z_2|^2 + |z_3|^2 = 1$, with $z_i \in \mathbb{C}$, and define the phases of the z_i to be ϕ_i.
- A choice of the coset representative in $SO(6)$ for points in S^5 is $x = e^{i\phi_1 Q_{12}} e^{i\phi_2 Q_{34}} e^{i\phi_3 Q_{56}} e^{i\psi Q_{35}} e^{i\theta Q_{13}}$, where Q's are the generators of $SO(6)$.
- We consider the quotient $\phi_1 \mapsto \phi_1 + \beta$, which we call the thermal quotient.
- The action of the thermal quotient on this coset representative is given by $\gamma : x \mapsto e^{i\beta Q_{12}} \cdot x$
- The thermal section is then given by

$$\sigma(x) = x.$$

The Traced Heat Kernel on Quotients of Symmetric Spaces

- We shall calculate the expression

$$\sum_{m\in\mathbb{Z}_N}\int_{\Gamma\backslash G/H} d\mu\left(x\right) K^{(S)}\left(x,\gamma^m\left(x\right);t\right),$$

where $d\mu\left(x\right)$ is the invariant measure on G/H.

- The integral over $\Gamma\backslash G/H$ can be replaced by the same integral over G/H with the multiplication of an appropriate volume factor. Therefore we shall instead calculate

$$\sum_{m\in\mathbb{Z}_N}\int_{G/H} d\mu\left(x\right) K^{(S)}\left(x,\gamma^m\left(x\right);t\right)$$

- With our choice of section, this implies that we shall need to evaluate the integral

$$\mathcal{I}_m = \int_{G/H} d\mu(x)\, \mathrm{Tr}_S\left(\left(x^{-1}\gamma^m x\right)^{(R)}\right),$$

in order to evaluate the heat kernel. Here $g^{(R)}$ is shorthand for $\mathcal{U}^{(R)}(g)$, $g \in G$.

- This integral over G/H can be lifted to an integral over G, and the trace over the representation S of H, which is a subspace of R, can be lifted to a trace over R.
- Using these simplifications, we can write

$$\mathcal{I}_m = \frac{d_S}{d_R} V_{G/H} \chi_R(\gamma^m).$$

A Sketch of the Proof

- As outlined above,

$$\begin{aligned}\mathcal{I}_m &= \int_{G/H} d\mu(x)\, \mathrm{Tr}_S\left(\left(x^{-1}\gamma^m x\right)^{(R)}\right)\\ &\sim \int_G dg\, \mathrm{Tr}_R\left(\left(g^{-1}\gamma^m g\right)^{(R)}\right) \sim \mathrm{Tr}_R\left(\gamma^m\right).\end{aligned}$$

- Now, $\gamma \sim e^{\theta_i H_i}$, where the H_is are the Cartans of $SO(6)$. This trace is then the character in the representation R.
- We therefore have

$$\mathcal{I}_m = \frac{d_S}{d_R} V_{G/H} \chi_R\left(\gamma^m\right),$$

where $\chi_R\left(\gamma^m\right)$ is the character in the representation R.

The traced Heat Kernel over Thermal S^5

- For the thermal quotient, the volume factor is just $\frac{\beta}{2\pi}$. Carrying through the above procedure gives us the expression for the traced heat kernel over thermal S^5

$$
\begin{aligned}
K^{(S)} &\equiv \sum_{m\in\mathbb{Z}_N}\int_{\Gamma\backslash G/H} d\mu\left(x\right) K^{(S)}\left(x,\gamma^{m}\left(x\right);t\right) \\
&= \frac{\beta}{2\pi}\sum_{m\in\mathbb{Z}_N}\sum_{R}\chi_{R}\left(\gamma^{m}\right)e^{tE_R^{(S)}}.
\end{aligned}
$$

- This is an expression purely in terms of the characters of Γ embedded in $SO(6)$ and the quadratic Casimirs of $SO(6)$ and $SO(5)$, both of which are known.
- We shall analytically continue this expression to obtain the traced heat kernel over AdS_5.

The Analytic Continuation to AdS_5

- The analytic continuation of the traced heat kernel is implemented by
    - replacing the $SO(6)$ character by the $SO(5,1)$ character, and
    - analytically continuing the sum over UIRs of $SO(6)$ to a sum over UIRs of $SO(5,1)$.
- A UIR of $SO(6)$ is labelled by (half-)integers (m_1, m_2, m_3) with $m_1 \geq m_2 \geq |m_3|$.
- The analytic continuation is:

$$\boxed{m_1 \mapsto i\lambda - 2, \quad \lambda \in \mathbb{R}_+.}$$

- This is an analytic continuation to the principal series of $SO(5,1)$ representations. For odd dimensional spacetimes, these are the only ones that matter for us.

The Traced Heat Kernel on Thermal AdS_5

- The traced heat kernel may be obtained from analytically continuing the expression for the traced heat kernel over S^5.

$$K^{(S)} = \frac{\beta}{2\pi} \sum_{m \in \mathbb{Z}_N} \sum_{R} \chi_R\left(\gamma^m\right) e^{tE_R^{(S)}}.$$

- The sum over R above is an implicit sum over m_1, m_2, m_3. The sum over m_1 gets analytically continued to an integral over λ and the other sums stay.
- We therefore obtain

$$\boxed{K^{(S)}\left(\gamma, t\right) = \frac{\beta}{2\pi} \sum_{m \in \mathbb{Z}} \sum_{m_2, m_3} \int_0^\infty d\lambda\, \chi_R\left(\gamma^m\right) e^{tE_R^{(S)}}}$$

as the expression for the traced heat kernel over thermal AdS_5.

- The eigenvalue of the Laplacian is fixed by the analytic continuation to be

$$\boxed{E_R^{(S)} = -\left(\lambda^2 + \zeta\right),}$$

  where ζ is a constant independent of λ and is a function of the representation S of H and of m_2, m_3.
- The thermal quotient acts by

$$\boxed{t \mapsto t + \beta}$$

- The character $\chi_R\left(\gamma^m\right)$ then has a very simple form

$$\boxed{\chi_R\left(\gamma^m\right) = \frac{1}{8}\frac{\cos m\beta\lambda}{\left(\sinh\frac{\beta}{2}\right)^4} d_{m_2,m_3},}$$

  where d_{m_2,m_3} is the dimension of the UIR of $SO(4)$ labelled by the highest weight (m_2, m_3).

- The integral over λ is therefore a gaussian integral and can be performed to obtain

$$K^{(S)}(\beta, t) = \frac{\beta}{32\pi} \sum_{m\in\mathbb{Z}} \sum_{m_2, m_3} d_{m_2,m_3} \frac{1}{\left(sinh\frac{m\beta}{2}\right)^4} \sqrt{\frac{\pi}{t}} e^{-\frac{m^2\beta^2}{4t} - t\zeta}$$

- This expression for the heat kernel also yields the expression for the one-loop partition function

$$logZ^{1-loop}_{(S)} = \frac{1}{16} \sum_{m\in\mathbb{Z}_+} \sum_{m_2, m_3} \frac{d_{m_2,m_3}}{m} \frac{1}{\left(sinh\frac{m\beta}{2}\right)^4} e^{-m\beta\sqrt{\zeta+m_R^2}},$$

where we have subtracted out the divergence at one-loop by dropping the $m = 0$ term.

Example: Partition Function of a Scalar on AdS_5

- For a scalar, $m_2 = m_3 = 0$ and $d_{m_2,m_3} = 1$, and $\sqrt{\zeta + m_R^2} = \Delta - 2$, where Δ is the conformal dimension of the scalar.
- The one-loop determinant is, therefore

$$logZ^{1-loop} = \sum_{m \in \mathbb{Z}_+} \frac{1}{m\left(1 - e^{-m\beta}\right)^4} e^{-m\beta\Delta}$$

- This sum can be evaluated to obtain

$$\boxed{logZ = -\sum_{n=0}^{\infty} \frac{(n+1)(n+2)(n+3)}{6} log\left(1 - e^{\beta(\Delta+n)}\right).}$$

- This matches with the result obtained using existing methods.

Denef, Hartnoll, Sachdev, arxiv:0908.2657

Summary

- The heat kernel on a general manifold is hard to evaluate.
- The group theoretic properties of symmetric spaces are of great use in carrying out this evaluation.
- We have calculated the heat kernel for an arbitrary-spin particle on thermal AdS_5.
- Our method is directly extendable to thermal AdS_{2n+1}.
- We are now in a position to go forward with the program outlined in the beginning.

Thank You